

LATTICES, GRAPHS, AND CONWAY MUTATION

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ABSTRACT. The d -invariant of an integral, positive definite lattice Λ records the minimal norm of a characteristic covector in each equivalence class $(\text{mod } 2\Lambda)$. We prove that the 2-isomorphism type of a connected graph is determined by the d -invariant of its lattice of integral cuts (or flows). As an application, we prove that a reduced, alternating link diagram is determined up to mutation by the Heegaard Floer homology of the link's branched double-cover. Thus, alternating links with homeomorphic branched double-covers are mutants.

1. INTRODUCTION.

Conway mutation has been in the news a lot lately. Given a sphere S^2 that meets a link $L \subset S^3$ transversely in four points, cut along it and reglue by an involution that fixes a pair of points disjoint from L and permutes $S^2 \cap L$. This process results in a new link $L' \subset S^3$, and a pair of links are called *mutants* if they are related by a sequence of such transformations. An analogous definition of Conway mutation applies to link diagrams.

A fundamental question about any link invariant is whether it can distinguish mutants. One such invariant is the homeomorphism type of the space $\Sigma(L)$, the double-cover of S^3 branched along L . As first noted by Viro, mutant links possess homeomorphic branched double-covers [Vir76, Thm.1], [Kaw96, Prop.3.8.2]. It follows that any invariant of branched double-covers will not distinguish mutants either. However, non-mutant links can possess homeomorphic branched double-covers, such as the pretzel knot $P(-2, 3, 7)$ and the torus knot $T(3, 7)$. It remains an intriguing open problem to classify distinct links with homeomorphic branched double-covers [Kir10, Probs.1.22&3.25].

Our purpose here is to show that within the class of *alternating* links, the Heegaard Floer homology of the branched double-cover provides a *complete* invariant for the mutation type.

Theorem 1.1. *Given a pair of connected, reduced alternating diagrams D, D' for a pair of links L, L' , the following assertions are equivalent:*

- (1) D and D' are mutants;
- (2) L and L' are mutants;
- (3) $\Sigma(L) \cong \Sigma(L')$; and
- (4) $\widehat{HF}(\Sigma(L)) \cong \widehat{HF}(\Sigma(L'))$, as absolutely graded, relatively spin^c -graded groups.

We derive Theorem 1.1 from Theorem 1.2 below, a combinatorial result. We proceed to sketch the main line of argument and then discuss some repercussions of Theorem 1.1.

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1.1. From topology to combinatorics. As indicated in the abstract, the focus of the paper is primarily combinatorial in nature. This is thanks to a description of the invariant $\widehat{HF}(\Sigma(L))$ for an alternating link L due to Ozsváth-Szabó, which we quickly review.

First, the manifold $\Sigma(L)$ is an L-space. This means that the invariant \widehat{HF} has rank one in each spin^c structure on $\Sigma(L)$, the set of which forms a torsor over $H^2(\Sigma(L); \mathbb{Z})$. Thus, the invariant is completely captured by its Heegaard Floer d -invariant, which for the case at hand is the mapping $d : \text{Spin}^c(\Sigma(L)) \rightarrow \mathbb{Q}$ that records the absolute grading in which each group is supported.

To express the d -invariant, choose a reduced, alternating diagram D for L , and let G denote its Tait graph. Associated to G is its lattice of integral flows $\mathcal{F}(G)$; this lattice is presented by the Goeritz matrix for D . For an integral lattice Λ , define the characteristic coset

$$\text{Char}(\Lambda) = \{\chi \in \Lambda^* \mid \langle \chi, y \rangle \equiv |y| \pmod{2}, \forall y \in \Lambda\},$$

where $\langle \cdot, \cdot \rangle$ denotes the inner product and $|\cdot|$ the norm (self-pairing) of an element. The set $C(\Lambda) = \text{Char}(\Lambda) \pmod{2\Lambda}$ forms a torsor over the discriminant group Λ^*/Λ . Given $[\chi] \in C(\Lambda)$, define

$$d_\Lambda([\chi]) = \min \left\{ \frac{|\chi'| - \text{rk}(\Lambda)}{4} \mid \chi' \in [\chi] \right\},$$

and call an element $\chi \in \text{Char}(\Lambda)$ *short* if its norm is minimal in $[\chi]$. We call the pair $(C(\Lambda), d_\Lambda)$ the d -invariant of the lattice Λ .

There exists a natural identification between the torsors $\text{Spin}^c(\Sigma(L))$ and $C(\mathcal{F}(G))$, and Ozsváth-Szabó showed that this identification extends to an isomorphism between the pairs $(\text{Spin}^c(\Sigma(L)), d)$ and $(C(\mathcal{F}(G)), -d_\mathcal{F})$ (Theorem 4.7). In summary, the isomorphism type of $\widehat{HF}(\Sigma(L))$ is determined by the (lattice theoretic) d -invariant of the lattice of integral flows on the Tait graph.

The foregoing description of $\widehat{HF}(\Sigma(L))$ begs the question: when do the flow lattices attached to a pair of graphs have isomorphic d -invariants? Our main combinatorial result answers this question.

Theorem 1.2. *The following are equivalent for a pair of 2-edge-connected graphs G, G' :*

- (1) $\mathcal{F}(G)$ and $\mathcal{F}(G')$ have isomorphic d -invariants;
- (2) $\mathcal{F}(G) \cong \mathcal{F}(G')$; and
- (3) G and G' are 2-isomorphic.

A 2-isomorphism between a pair of 2-edge-connected graphs is a cycle-preserving bijection between their edge sets. Note that Theorem 1.2 applies to arbitrary 2-edge-connected graphs, not just planar ones. Also, the implication (3) \implies (2) appears in [BdlHN97, Prop.5], and (2) \implies (3) resolves the question implicit at the end of that paper. An extended version of Theorem 1.2 appears as Theorem 3.8 below.

1.2. Prospectus on Theorem 1.1. We sketch the proof of Theorem 1.1, using Theorem 1.2. The forward implications in Theorem 1.1 are immediate, so it stands to establish (4) \implies (1). Thus, choose a pair of reduced, alternating diagrams D, D' for a pair of links L, L' for which

$\widehat{HF}(\Sigma(L)) \cong \widehat{HF}(\Sigma(L'))$. It follows that $\mathcal{F}(G)$ and $\mathcal{F}(G')$ have isomorphic d -invariants, where G, G' denote the Tait graphs. By Theorem 1.2, it follows that G and G' are 2-isomorphic. Now we invoke two graph theoretic results. First, a theorem of Whitney asserts that a pair of 2-isomorphic graphs are related by a sequence of switches. Second, using Whitney's result, a theorem of Mohar-Thomassen about planar graphs asserts that any two planar drawings of a pair of 2-isomorphic planar graphs are related by a sequence of flips, planar switches, and swaps. Each of these transformations of planar graphs corresponds to a Conway mutation of link diagrams, so it follows that D, D' are mutants.

1.3. Prospectus on Theorem 1.2. Now we sketch the proof of Theorem 1.2. To a graph G we associate the chain group $C_1(G; \mathbb{Z})$. This group naturally inherits the structure of a lattice by taking the edge set of G as an orthonormal basis. Within $C_1(G; \mathbb{Z})$ sits a pair of distinguished sublattices, the lattice of integral cuts $\mathcal{C}(G)$, and the aforementioned lattice of integral flows $\mathcal{F}(G)$. (For the case of a planar graph G with planar dual G^* , we have $\mathcal{C}(G) \cong \mathcal{F}(G^*)$.) In general, $\mathcal{C}(G)$ and $\mathcal{F}(G)$ are complementary, primitive sublattices of $C_1(G; \mathbb{Z})$. Furthermore, every short characteristic covector for $\mathcal{C}(G)$ and $\mathcal{F}(G)$ is the restriction of one for $C_1(G; \mathbb{Z})$. It follows that the d -invariants of these sublattices are opposite one another: that is, there exists a natural isomorphism $(C(\mathcal{F}(G)), d_{\mathcal{F}}) \xrightarrow{\sim} (C(\mathcal{C}(G)), -d_{\mathcal{C}})$.

Now suppose that the flow lattices of G and G' have isomorphic d -invariants. Since the discriminant groups are isomorphic, we can glue $\mathcal{F}(G)$ and $\mathcal{C}(G')$ to produce an integral, positive definite, unimodular lattice Λ . Furthermore, since they have opposite d -invariants, Λ has vanishing d -invariant. By a theorem of Elkies, it follows that Λ admits an orthonormal basis. Using the fact that every short characteristic covector for $\mathcal{F}(G)$ and $\mathcal{C}(G')$ is the restriction of one for Λ , we can set the orthonormal basis for Λ in one-to-one correspondence with the edge sets of G and G' . It easily follows that the resulting bijection between the edge sets of G and G' is a 2-isomorphism.

1.4. Repercussions of Theorem 1.1. Since diagrammatic mutations clearly preserve the number of crossings, Theorem 1.1 implies that two reduced, alternating diagrams for the *same* link have the same number of crossings. Furthermore, if a reduced, alternating diagram admits no non-trivial mutation, then it is the unique reduced, alternating diagram representing its link type (cf. [Sch93]). Of course, much more is known now: any minimal crossing diagram of an alternating link is alternating [Kau87, Mur87, Thi87], and any two such diagrams are related by a sequence of *flypes* [MT91]. We simply point out that we obtain these corollaries in a rather different manner from how they were originally deduced, using graphs, lattices, and Floer homology in place of the Jones polynomial and explicit geometric arguments.

Second, Theorem 1.1 generalizes the homeomorphism classification of the three-dimensional lens spaces and the isotopy classification of two-bridge links. The lens spaces arise as the branched double-covers of the two-bridge links, and we argue directly in Proposition 4.6 that a pair of two-bridge diagrams in standard position are mutants iff they coincide up to isotopy and reversal. Using this fact, Theorem 1.1 implies a one-to-one correspondence between such diagrams and lens spaces, yielding at once the classification of both. We point out that \widehat{HF} recovers the Reidemeister torsion by a result of Rustamov [Rus05, Thm.3.4], which is

well-known to completely distinguish the homeomorphism types of lens spaces [Bro60, Rei35], and which leads to the classification of 2-bridge links [Sch56]. Of course, the homeomorphism classification follows from the stronger result that every lens space possesses a unique Heegaard torus up to isotopy [Bon83, Thm.1], [HR85, Thm.5.1]. We simply point out, once again, the different manner of our argument.

Third, note that Theorem 1.1 cannot extend too far beyond the domain of alternating links, due to the existence of non-mutant links with homeomorphic branched double-covers. It would be interesting to know whether Theorem 1.1 generalizes to quasi-alternating links. Note that the invariant \widehat{HF} does not distinguish the branched double-covers of alternating and non-alternating knots in general, such as the unknot and $T(3, 5) \# \overline{T(3, 5)}$. However, we propose the following conjecture.

Conjecture 1.3. *There does not exist a pair of links, one alternating, the other non-alternating, with homeomorphic branched double-covers.*

We provide some limited evidence in support of Conjecture 1.3. First, no mutant pair violates Conjecture 1.3, since a result of Menasco implies that mutation preserves alternatingness [Men84, Proof of Thm.3(b)]. Second, Hodgson-Rubinstein showed that a two-bridge link is uniquely determined by its branched double-cover [HR85, Cor.4.12]. Third, Dunfield investigated knots with at most 16 crossings whose branched double-covers are hyperbolic and of small enough volume to appear in the Hodgson-Weeks census of closed hyperbolic 3-manifolds. He reports 3765 non-alternating knots with such branched covers but only 178 alternating knots, and no manifold appears as the branched double-cover of both kinds of knots [Dun10]. Finally, and most persuasively to this author, is the lack of any counterexample known to the (non-exhaustive) list of experts we consulted!

1.5. Organization. The main body is organized into three sections: *Lattices*, *Graphs*, and *Conway Mutation*. Each section begins at a basic level and invokes a key auxiliary result: Elkies’s theorem on unimodular lattices; Whitney’s theorem on 2-isomorphism of graphs; and Ozsváth-Szabó’s theorem on $\widehat{HF}(\Sigma(L))$, respectively. It remains a curious fact that although Elkies’s theorem asserts a purely algebraic fact and we use it towards a combinatorial end, its only known proof relies on analytical methods (modular forms). Lastly, it is perhaps fitting that the chief insight involved the use of lattice gluing, a technique we learned from Conway, to establish the desired result about Conway mutation of alternating links.

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2. LATTICES.

2.1. Preparation. A *lattice* consists of a finitely-generated, free abelian group Λ equipped with a non-degenerate, symmetric bilinear form

$$\langle \cdot, \cdot \rangle : \Lambda \times \Lambda \rightarrow \mathbb{Q}.$$

It is *integral* if the image of its pairing lies in \mathbb{Z} , and the symbol Λ will always denote an integral lattice in what follows. The form extends to a \mathbb{Q} -valued pairing on $\Lambda \otimes \mathbb{Q}$, which allows us to define the *dual lattice*

$$\Lambda^* := \{x \in \Lambda \otimes \mathbb{Q} \mid \langle x, y \rangle \in \mathbb{Z}, \forall y \in \Lambda\}.$$

Given $x \in \Lambda^*$, denote by \bar{x} its image in the *discriminant group* $\bar{\Lambda} := \Lambda^*/\Lambda$. The *discriminant* $\text{disc}(\Lambda)$ is the order of this finite group, and Λ is *unimodular* if $\text{disc}(\Lambda) = 1$. For example, the lattice generated by n orthonormal elements is the integral, unimodular lattice denoted \mathbb{Z}^n . The pairing on Λ descends to a non-degenerate, symmetric bilinear form

$$b : \bar{\Lambda} \times \bar{\Lambda} \rightarrow \mathbb{Q}/\mathbb{Z},$$

$$b(\bar{x}, \bar{y}) \equiv \langle x, y \rangle \pmod{1},$$

the *discriminant form* (or, for topologists, the *linking form*). The *norm* of $x \in \Lambda^*$ is the self-pairing $|x| := \langle x, x \rangle$, and we set $q(\bar{x}) := b(\bar{x}, \bar{x})$.

2.2. The characteristic coset. Let

$$\text{Char}(\Lambda) = \{\chi \in \Lambda^* \mid \langle \chi, y \rangle \equiv |y| \pmod{2}, \forall y \in \Lambda\}$$

denote the set of *characteristic covectors* for Λ . This set constitutes a distinguished coset in $\Lambda^*/2\Lambda^*$. Correspondingly, the set

$$C(\Lambda) := \text{Char}(\Lambda) \pmod{2\Lambda}$$

forms a torsor over the group $2\Lambda^*/2\Lambda \cong \bar{\Lambda}$. Thus, for a unimodular lattice, such as \mathbb{Z}^n , this torsor has one element. Given $\chi \in \text{Char}(\Lambda)$, let $[\chi]$ denote its image in $C(\Lambda)$. We never refer to $\bar{\chi} \in \bar{\Lambda}$, so no confusion should result. We obtain a map

$$\rho : C(\Lambda) \rightarrow \mathbb{Q}/2\mathbb{Z},$$

$$\rho([\chi]) \equiv \frac{|\chi| - \sigma(\Lambda)}{4} \pmod{2},$$

where $\sigma(\Lambda)$ denotes the *signature* of the pairing on Λ (cf. [OS05], [OSz03, §1.1]). The map ρ is well-defined since

$$\frac{1}{4}(|\chi + 2y| - |\chi|) = \langle \chi, y \rangle + |y| \equiv 0 \pmod{2}, \quad \forall y \in \Lambda.$$

By definition, the pair $(C(\Lambda), \rho)$ is the ρ -invariant of Λ . In this terminology, we have the following classical result.

Theorem 2.1 (van der Blij [vdB59]). *The mapping $\rho : C(\Lambda) \rightarrow \mathbb{Q}/2\mathbb{Z}$ vanishes for a unimodular lattice Λ .* \square

Definition 2.2. *Let $f_i : C_i \rightarrow S$ be a map from a G_i -torsor C_i to a set S , for $i = 1, 2$. An isomorphism*

$$\varphi : (C_1, f_1) \xrightarrow{\sim} (C_2, f_2)$$

consists of a bijection $\varphi : C_1 \rightarrow C_2$ and a group isomorphism $\varphi : G_1 \rightarrow G_2$ such that $f_2 = \varphi \circ f_1$ and $\varphi(c) - \varphi(c') = \varphi(c - c')$ for all $c, c' \in C_1$.

Lemma 2.3 (cf. [OS05], Prop.6). *An isomorphism of ρ -invariants $\varphi : (C(\Lambda_1), \rho_1) \rightarrow (C(\Lambda_2), \rho_2)$ induces an isomorphism of discriminant forms $\varphi : (\bar{\Lambda}_1, b_1) \rightarrow (\bar{\Lambda}_2, b_2)$.*

Proof. Given a lattice Λ , fix $\chi \in \text{Char}(\Lambda)$ and select $x, y \in \Lambda^*$. From the identity

$$\langle x, y \rangle = \frac{1}{8}(|\chi + 2x + 2y| + |\chi| - |\chi + 2x| + |\chi + 2y|)$$

we obtain

$$b(\bar{x}, \bar{y}) \equiv \frac{1}{2}(\rho([\chi] + 2\bar{x} + 2\bar{y}) + \rho([\chi]) - \rho([\chi] + 2\bar{x}) - \rho([\chi] + 2\bar{y})) \pmod{1}.$$

The statement of the Lemma now follows directly. □

2.3. Gluing. Given a sublattice Λ_1 of a lattice Λ , we obtain a natural restriction map

$$r_1 : \Lambda^* \rightarrow \Lambda_1^*.$$

The sublattice $\Lambda_1 \subset \Lambda$ is *primitive* if the mapping r_1 surjects. A pair of sublattices $\Lambda_1, \Lambda_2 \subset \Lambda$ are *complementary* if they are orthogonal and their ranks sum to that of Λ . One may check that $\Lambda_1, \Lambda_2 \subset \Lambda$ are primitive and complementary iff $\Lambda_1^\perp = \Lambda_2$ and $\Lambda_2^\perp = \Lambda_1$, although we shall not need this fact. Versions of the following gluing Lemma have been observed by a number of researchers, e.g. [CS99, Ch.4, Thm.1], [OS05, Prop.1].

Lemma 2.4. *Suppose that Λ_1, Λ_2 are a pair of complementary, primitive sublattices of a unimodular lattice Λ . Then there exists a natural isomorphism*

$$(1) \quad \varphi : (\bar{\Lambda}_1, b_1) \xrightarrow{\sim} (\bar{\Lambda}_2, -b_2).$$

Conversely, suppose that Λ_1 and Λ_2 are a pair of integral lattices and there exists an isomorphism φ as in (1). Then the glue lattice

$$\Lambda_1 \oplus_\varphi \Lambda_2 := \{x + y \in \Lambda_1^* \oplus \Lambda_2^* \mid \varphi(\bar{x}) = \bar{y}\}$$

is an integral, unimodular lattice that contains Λ_1 and Λ_2 as complementary, primitive sublattices.

Proof. (\implies) To quote [CS99], the isomorphism is given by $\varphi(\bar{x}) = \bar{y}$ whenever $x + y \in \Lambda$. Verification of the stated properties is straightforward.

(\impliedby) By construction, $\Lambda := \Lambda_1 \oplus_\varphi \Lambda_2$ is a lattice that contains Λ_1 and Λ_2 as complementary sublattices. It is integral since, given $x + y \in \Lambda$, we have

$$|x + y| = |x| + |y| \equiv q_1(\bar{x}) + q_2(\bar{y}) \equiv q_1(\bar{x}) + q_2(\varphi(\bar{x})) \equiv q_1(\bar{x}) - q_1(\bar{x}) \equiv 0 \pmod{1}.$$

It is unimodular since $\Lambda/(\Lambda_1 \oplus \Lambda_2) \cong \{(\bar{x}, \varphi(\bar{x})) \in \bar{\Lambda}_1 \oplus \bar{\Lambda}_2\}$ is a square-root order subgroup of $\bar{\Lambda}_1 \oplus \bar{\Lambda}_2 \cong (\Lambda_1 \oplus \Lambda_2)^*/(\Lambda_1 \oplus \Lambda_2)$. Since Λ is integral and unimodular, the restriction maps r_1, r_2 are simply the projections $\Lambda \rightarrow \Lambda_1^*, \Lambda \rightarrow \Lambda_2^*$. These maps surject by construction, so Λ_1 and Λ_2 are primitive sublattices of Λ . □

The construction of Lemma 2.4 behaves well with respect to characteristic cosets.

Lemma 2.5. *Suppose that Λ_1, Λ_2 are a pair of complementary, primitive sublattices of a unimodular lattice Λ . Then there exists a natural isomorphism*

$$(2) \quad \varphi : (C(\Lambda_1), \rho_1) \xrightarrow{\sim} (C(\Lambda_2), -\rho_2).$$

Conversely, suppose that Λ_1 and Λ_2 are a pair of integral lattices and there exists an isomorphism φ as in (2). Then $\Lambda_1 \oplus_\varphi \Lambda_2$ is an integral, unimodular lattice that contains Λ_1 and Λ_2 as complementary, primitive sublattices.

The mapping of discriminant groups used in the gluing $\Lambda_1 \oplus_\varphi \Lambda_2$ is the one implicit in the torsor map φ , as in Definition 2.2.

Proof. (\implies) To begin with, observe that

$$(3) \quad 2\Lambda \cap (2\Lambda_1 + \Lambda_2^*) = 2\Lambda_1 + 2\Lambda_2$$

(we identify $\Lambda_1^\perp \subset \Lambda$ with Λ_2^* using Lemma 2.4). Indeed, if $2(x + y) \in 2L$ with $2x \in 2\Lambda_1$, $2y \in \Lambda_2^*$, then $0 = \bar{x} \in \bar{\Lambda}_1$, so $0 = \varphi(\bar{x}) = \bar{y} \in \bar{\Lambda}_2$ using the isomorphism of Lemma 2.4. Thus, $y \in \Lambda_2$ and $2y \in 2\Lambda_2$, as desired.

Next, each restriction map r_i clearly carries $\text{Char}(\Lambda)$ onto a subset of $\text{Char}(\Lambda_i)$. Furthermore, since r_i maps Λ onto Λ_i^* , it carries 2Λ onto $2\Lambda_i^*$, and hence the coset $\text{Char}(\Lambda)$ onto $\text{Char}(\Lambda_i)$. Thus, given a pair of elements $\chi_1, \chi'_1 \in \text{Char}(\Lambda_1)$ with $[\chi_1] = [\chi'_1] \in C(\Lambda_1)$, there exists a pair of elements $\chi_2, \chi'_2 \in \text{Char}(\Lambda_2)$ such that $\chi = \chi_1 + \chi_2$ and $\chi' = \chi'_1 + \chi'_2$ belong to $\text{Char}(\Lambda)$. Their difference $\chi - \chi'$ belongs to 2Λ since Λ is unimodular, and $\chi_1 - \chi'_1 \in 2\Lambda_1$ by assumption. It follows from (3) that $[\chi_2] = [\chi'_2] \in C(\Lambda_2)$. Thus, we obtain a well-defined mapping

$$\begin{aligned} \varphi : C(\Lambda_1) &\xrightarrow{\sim} C(\Lambda_2), \\ \varphi([\chi_1]) &= [\chi_2] \iff \chi_1 + \chi_2 \in \text{Char}(\Lambda), \end{aligned}$$

whence

$$\text{Char}(\Lambda) = \{\chi_1 + \chi_2 \in \text{Char}(\Lambda_1) \oplus \text{Char}(\Lambda_2) \mid \varphi([\chi_1]) = [\chi_2]\}.$$

Furthermore, if $\varphi([\chi_1]) = [\chi_2]$, then

$$\rho([\chi_1]) + \rho([\chi_2]) \equiv \frac{|\chi_1| - \sigma(\Lambda_1)}{4} + \frac{|\chi_2| - \sigma(\Lambda_2)}{4} = \frac{|\chi_1 + \chi_2| - \sigma(\Lambda)}{4} \equiv \rho(\Lambda) \equiv 0 \pmod{2},$$

applying Theorem 2.1 at the last step. Finally, the mapping φ covers the isomorphism of discriminant forms from Lemma 2.4, so it preserves the torsor structure. This establishes the first part of the Lemma.

(\impliedby) This follows directly on combination of Lemma 2.3 and the second part of Lemma 2.4. \square

2.4. The positive definite case. When the form \langle, \rangle on Λ is *positive definite*, its rank n equals its signature $\sigma(\Lambda)$, and we obtain a \mathbb{Q} -valued lift of the ρ -invariant by defining

$$\begin{aligned} d : C(\Lambda) &\rightarrow \mathbb{Q}, \\ d([\chi]) &= \min \left\{ \frac{|\chi'| - n}{4} \mid \chi' \in [\chi] \right\}. \end{aligned}$$

By definition, the pair $(C(\Lambda), d)$ is the d -invariant $d(\Lambda)$ of the positive definite lattice Λ . It is clearly additive, in the sense that there exists a natural isomorphism

$$(C(\Lambda), d) \xrightarrow{\sim} (C(\Lambda_1) \oplus C(\Lambda_2), d_1 + d_2)$$

whenever $\Lambda \cong \Lambda_1 \oplus \Lambda_2$. We further define

$$\text{Short}(\Lambda) = \{\chi \in \text{Char}(\Lambda) \mid |\chi| \leq |\chi'|, \forall \chi' \in [\chi]\},$$

and refer to elements of $\text{Short}(\Lambda)$ as *short* characteristic covectors. For example,

$$\text{Short}(\mathbb{Z}^n) = \{\chi \mid \langle \chi, e_i \rangle = \pm 1 \forall i\},$$

where $\{e_1, \dots, e_n\}$ denotes an orthonormal basis for \mathbb{Z}^n . Thus, $|\chi| = n$ for all $\chi \in \text{Short}(\mathbb{Z}^n)$, so the mapping $d : C(\mathbb{Z}^n) \rightarrow \mathbb{Q}$ is zero. Conversely, we have the following fundamental result.

Theorem 2.6 (Elkies [Elk95]). *If Λ is a rank n , unimodular, integral, positive definite lattice and $|\chi| \geq n$ for all $\chi \in \text{Char}(\Lambda)$, then $\Lambda \cong \mathbb{Z}^n$, i.e. Λ admits an orthonormal basis.* \square

Observe that in the construction of §2.3, each map r_i restricts to a map

$$(4) \quad \text{Short}(\Lambda) \rightarrow \text{Short}(\Lambda_i),$$

where $\Lambda = \Lambda_1 \oplus_{\varphi} \Lambda_2$. Indeed, given $\chi_1 + \chi_2 \in \text{Short}(\Lambda)$ and any $\chi'_i \in \text{Char}(\Lambda_i)$ with $\chi'_i \in [\chi_i]$, $i = 1, 2$, we have $\chi'_1 + \chi'_2 \in \text{Char}(\Lambda)$, and

$$|\chi'_1| + |\chi'_2| = |\chi'_1 + \chi'_2| \geq |\chi_1 + \chi_2| = |\chi_1| + |\chi_2|,$$

which shows that $|\chi'_i| \geq |\chi_i|$, $i = 1, 2$. In general, the restriction (4) need not surject. For example, if

$$\Lambda_1 = \text{span}(e_1 + 2e_2), \Lambda_2 = \text{span}(-2e_1 + e_2) \subset \mathbb{Z}^2,$$

then $|\text{Short}(\mathbb{Z}^2)| = 4$, whereas $|\text{Short}(\Lambda_1)| \geq |C(\Lambda_1)| = \text{disc}(\Lambda_1) = 5$. However, it is clear that $\text{Short}(\Lambda) \rightarrow \text{Short}(\Lambda_1)$ surjects iff $\text{Short}(\Lambda) \rightarrow \text{Short}(\Lambda_2)$ does.

Using Elkies's Theorem, we obtain the following refinement of Lemma 2.5.

Proposition 2.7. *Suppose that Λ_1, Λ_2 are a pair of complementary, primitive sublattices of \mathbb{Z}^n , and the restriction $\text{Short}(\mathbb{Z}^n) \rightarrow \text{Short}(\Lambda_1)$ surjects. Then there exists a natural isomorphism*

$$(5) \quad \varphi : (C(\Lambda_1), d_1) \xrightarrow{\sim} (C(\Lambda_2), -d_2).$$

Conversely, suppose that Λ_1 and Λ_2 are a pair of integral lattices and there exists an isomorphism as in (5). Then $\Lambda_1 \oplus_{\varphi} \Lambda_2 \cong \mathbb{Z}^n$, and the restrictions $\text{Short}(\mathbb{Z}^n) \rightarrow \text{Short}(\Lambda_i)$ surject.

Proof. Set $n_i = \text{rk}(\Lambda_i)$, $i = 1, 2$.

(\implies) We use the isomorphism φ of Lemma 2.5. Suppose that χ_1 is a short representative for its class in $C(\Lambda_1)$. Since $\text{Short}(\mathbb{Z}^n) \rightarrow \text{Short}(\Lambda_1)$ surjects, we have $\chi_1 + \chi_2 \in \text{Short}(\mathbb{Z}^n)$ for some $\chi_2 \in \text{Short}(\Lambda_2)$. Since

$$(6) \quad \frac{|\chi_1 + \chi_2| - n}{4} = \frac{|\chi_1| - n_1}{4} + \frac{|\chi_2| - n_2}{4} = d_1([\chi_1]) + d_2([\chi_2]) = d_1([\chi_1]) + d_2(\varphi([\chi_2]))$$

and the left-most term is zero, it follows that φ yields the desired isomorphism.

(\Leftarrow) Select any $\chi \in \text{Short}(\Lambda_1 \oplus_\varphi \Lambda_2)$. Then $\chi = \chi_1 + \chi_2$ with $\varphi([\chi_1]) = [\chi_2]$ and $\chi_i \in \text{Short}(\Lambda_i)$, $i = 1, 2$. Again, (6) holds, and now the right-most term is zero, so $|\chi| = n$ and $\Lambda_1 \oplus_\varphi \Lambda_2 \cong \mathbb{Z}^n$ by Theorem 2.6. Furthermore, if $\chi_1 \in \text{Short}(\Lambda_1)$ is given and $\chi_2 \in \text{Short}(\Lambda_2)$ is chosen so that $\varphi([\chi_1]) = [\chi_2]$, then (6) applies again to show that $\chi_1 + \chi_2 \in \text{Short}(\mathbb{Z}^n)$. Hence the restriction $\text{Short}(\mathbb{Z}^n) \rightarrow \text{Short}(\Lambda_1)$ surjects.

□

2.5. Rigid embeddings. The following Proposition establishes a condition under which a lattice admits an essentially unique embedding into \mathbb{Z}^n . It plays a key role in the proof of Theorem 3.8.

Proposition 2.8. *Let Λ denote a lattice, B_Λ a basis for Λ , Z_i a lattice with orthonormal basis B_i , and $\iota_i : \Lambda \hookrightarrow Z_i$ an embedding, $i = 1, 2$. Suppose that ι_1 has the property that*

$$(7) \quad \langle \iota_1(x), e \rangle \in \{0, 1\}, \quad \forall x \in B_\Lambda, e \in B_1,$$

and

$$(8) \quad \forall e \in B_1, \exists x \in B_\Lambda \text{ s.t. } \langle \iota_i(x), e \rangle \neq 0;$$

and suppose that both restriction maps $Z_i^* \rightarrow \Lambda^*$ induce surjections

$$r_i : \text{Short}(Z_i) \rightarrow \text{Short}(\Lambda).$$

Then there exists an embedding $\iota : Z_1 \hookrightarrow Z_2$ such that $\iota_2 = \iota \circ \iota_1$.

We work towards the proof of Proposition 2.8 through a sequence of Lemmas. Define

$$\text{supp}^\pm(x) = \{e \in B_i \mid \pm \langle x, e \rangle > 0\}, \quad \text{supp}(x) = \text{supp}^+(x) \cup \text{supp}^-(x), \quad \forall x \in Z_i, i = 1, 2.$$

Hence $\text{supp}^-(\iota_1(x)) = \emptyset$ and $|x| = |\text{supp}(\iota_1(x))|$, $\forall x \in B_\Lambda$. Note as well that

$$\langle \chi, \iota_i(x) \rangle = \langle r_i(\chi), x \rangle, \quad \forall x \in B_\Lambda, \chi \in \text{Short}(Z_i), i = 1, 2.$$

Given a subset $Y \subset B_\Lambda$, define

$$S(Y) = \{\chi \in \text{Short}(\Lambda) \mid \langle \chi, y \rangle = |y|, \forall y \in Y\}$$

and

$$S_i(Y) = \{\chi \in \text{Short}(Z_i) \mid \langle \chi, \iota_i(y) \rangle = |y|, \forall y \in Y\}, \quad i = 1, 2.$$

In particular, $S(\emptyset) = \text{Short}(\Lambda)$. Note, crucially, that since r_i surjects, we have

$$(9) \quad S(Y) = \{r_i(\chi) \mid \chi \in S_i(Y)\}, \quad i = 1, 2.$$

On the other hand, (7) implies that

$$S_1(Y) = \{\chi \in \text{Short}(Z_1) \mid \bigcup_{y \in Y} \text{supp}(\iota_1(y)) \subset \text{supp}^+(\chi)\}.$$

Given an element $x \in B_\Lambda$, define

$$M(x, Y) = \max\{\langle \chi, x \rangle \mid \chi \in S(Y)\}, \quad m(x, Y) = \min\{\langle \chi, x \rangle \mid \chi \in S(Y)\},$$

and

$$D(x, Y) = \frac{1}{2}(M(x, Y) - m(x, Y)).$$

Hence $M(x, Y)$ is attained by $r_1(\chi_0)$ and $m(x, Y)$ is attained by $r_1(\chi_Y)$, where $\chi_0, \chi_Y \in \text{Short}(Z_1)$ are defined by

$$\chi_0 = \sum_{e \in B_1} e, \quad \text{supp}^+(\chi_Y) = \bigcup_{y \in Y} \text{supp}(\iota_1(y)).$$

It follows that

$$(10) \quad D(x, Y) = |\text{supp}(\iota_1(x)) - \bigcup_{y \in Y} \text{supp}(\iota_1(y))|.$$

Lemma 2.9. *We have*

$$|\langle \iota_2(x), f \rangle| \in \{0, 1\}, \quad \forall x \in B_\Lambda, f \in B_2.$$

In particular,

$$(11) \quad |x| = |\text{supp}(\iota_1(x))| = |\text{supp}(\iota_2(x))|, \quad \forall x \in B_\Lambda.$$

Proof. Choose $x \in B_\Lambda$. By (10) we obtain

$$(12) \quad D(x, \emptyset) = |\text{supp}(\iota_1(x))| = |x| = |\iota_2(x)| = \sum_{f \in B_2} \langle \iota_2(x), f \rangle^2.$$

On the other hand, $M(x, \emptyset)$ and $m(x, \emptyset)$ are attained by $r_2(\chi_M)$ and $r_2(\chi_m)$, respectively, where $\chi_M, \chi_m \in \text{Short}(Z_2)$ are defined by

$$\text{supp}^+(\chi_M) = \text{supp}^+(\iota_2(x)), \quad \text{supp}^-(\chi_M) = \text{supp}^-(\iota_2(x)).$$

Hence

$$D(x, \emptyset) = \sum_{f \in B_2} |\langle \iota_2(x), f \rangle|.$$

Comparing with (12) and the inequality $|\langle \iota_2(x), f \rangle| \leq \langle \iota_2(x), f \rangle^2$ yields

$$|\langle \iota_2(x), f \rangle| = \langle \iota_2(x), f \rangle^2, \quad \forall f \in B_2,$$

from which the statement of the Lemma follows. □

Lemma 2.10. *We have*

$$\langle x, y \rangle = |\text{supp}(\iota_1(x)) \cap \text{supp}(\iota_1(y))| = |\text{supp}(\iota_2(x)) \cap \text{supp}(\iota_2(y))|, \quad \forall x, y \in B_\Lambda.$$

Proof. Choose $x, y \in B_\Lambda$. We obtain

$$S_2(\{y\}) = \{\chi \in \text{Short}(Z_2) \mid \text{supp}^\pm(y) \subset \text{supp}^\pm(\chi)\}.$$

It follows that $M(x, \{y\})$ and $m(x, \{y\})$ are attained by $r_2(\chi_M)$ and $r_2(\chi_m)$, respectively, for the elements $\chi_M, \chi_m \in \text{Short}(Z_2)$ defined by

$$\begin{aligned} \text{supp}^+(\chi_M) &= \text{supp}^+(\iota_2(y)) \cup \text{supp}^+(\iota_2(x)) - \text{supp}^-(\iota_2(y)), \\ \text{supp}^+(\chi_m) &= \text{supp}^+(\iota_2(y)) \cup \text{supp}^-(\iota_2(x)) - \text{supp}^-(\iota_2(y)). \end{aligned}$$

We obtain

$$(13) \quad D(x, \{y\}) = |\text{supp}(\iota_2(x)) - \text{supp}(\iota_2(y))|.$$

The first equality in the Lemma follows from (7), while the second now follows on combination of (10), (11), and (13). \square

Lemma 2.11. *There exists an orthonormal basis B'_2 for Z_2 such that*

$$(14) \quad \langle \iota_2(x), f \rangle \in \{0, 1\}, \quad \forall x \in B_\Lambda, f \in B'_2.$$

Proof. Choose a pair of elements $x, y \in B_\Lambda$. The pairing $\langle \iota_2(x), \iota_2(y) \rangle$ is a sum of terms

$$(15) \quad \langle \iota_2(x), f \rangle \cdot \langle \iota_2(y), f \rangle, \quad f \in \text{supp}(\iota_2(x)) \cap \text{supp}(\iota_2(y)),$$

each of which is ± 1 . On the other hand, we have

$$\langle \iota_2(x), \iota_2(y) \rangle = \langle x, y \rangle = |\text{supp}(\iota_2(x)) \cap \text{supp}(\iota_2(y))|$$

by Lemma 2.10. It follows that each term (15) is $+1$, so

$$(16) \quad \langle \iota_2(x), f \rangle = \langle \iota_2(y), f \rangle, \quad \forall f \in \text{supp}(\iota_2(x)) \cap \text{supp}(\iota_2(y)).$$

For given a fixed $f \in B_2$, it follows from (16) that either $\langle x, f \rangle \geq 0, \forall x \in B_\Lambda$, or else $\langle x, f \rangle \leq 0, \forall x \in B_\Lambda$. In the first case (which includes the possibility that $\langle f, x \rangle = 0, \forall x \in B_\Lambda$), we declare $f \in B'_2$; otherwise, $-f \in B'_2$. The resulting orthonormal basis B'_2 clearly fulfills (14). \square

Proof of Proposition 2.8. Using the basis B'_2 of Lemma 2.11, we obtain

$$S_2(Y) = \{\chi \in \text{Short}(Z_2) \mid \bigcup_{y \in Y} \text{supp}(\iota_2(y)) \subset \text{supp}^+(\chi)\}.$$

Just as (10) follows from (7), it follows from (14) that

$$(17) \quad D(x, Y) = |\text{supp}(\iota_i(x)) - \bigcup_{y \in Y} \text{supp}(\iota_i(y))|, \quad i = 1, 2.$$

Now apply inclusion-exclusion to (17) to obtain, for all partitions $B_\Lambda = X \cup Y$ and $z \in X$,

$$\begin{aligned} \left| \bigcap_{x \in X} \text{supp}(\iota_1(x)) - \bigcup_{y \in Y} \text{supp}(\iota_1(y)) \right| &= \sum_{X' \subset X-z} (-1)^{|X'|} |\text{supp}(\iota_1(z)) - \bigcup_{y \in X' \cup Y} \text{supp}(\iota_1(y))| \\ &= \sum_{X' \subset X-z} (-1)^{|X'|} D(z, X' \cup Y) \\ &= \sum_{X' \subset X-z} (-1)^{|X'|} |\text{supp}(\iota_2(z)) - \bigcup_{y \in X' \cup Y} \text{supp}(\iota_2(y))| \\ &= \left| \bigcap_{x \in X} \text{supp}(\iota_2(x)) - \bigcup_{y \in Y} \text{supp}(\iota_2(y)) \right|. \end{aligned}$$

Thus, we can set each pair of *atoms* into one-to-one correspondence:

$$\iota_{X,Y} : \bigcap_{x \in X} \text{supp}(\iota_1(x)) - \bigcup_{y \in Y} \text{supp}(\iota_1(y)) \xrightarrow{\sim} \bigcap_{x \in X} \text{supp}(x) - \bigcup_{y \in Y} \text{supp}(y),$$

for all partitions $B_\Lambda = X \cup Y$. By (8), these atoms partition the sets B_1, B_2 , and piecing together all the various $\iota_{X,Y}$ yields a bijection

$$\iota : B_1 \xrightarrow{\sim} \bigcup_{x \in B_\Lambda} \text{supp}(\iota_2(x)) \subset B'_2$$

with the property that

$$\{\iota(e) \mid e \in \text{supp}(\iota_1(x))\} = \text{supp}(\iota_2(x)), \quad \forall x \in B_\Lambda.$$

Extend ι by linearity to a map $\iota : Z_1 \rightarrow Z_2$. It is clear that $\iota_2 = \iota \circ \iota_1$ since this relation holds for the basis B_Λ , and this establishes the Proposition. \square

3. GRAPHS.

3.1. The cut lattice and the flow lattice. (cf. [BdH97], [GR01, Ch.14]) Let $G = (V, E)$ denote a finite, loopless, undirected graph with vertex set $V = \{v_1, \dots, v_m\}$ and edge set $E = \{e_1, \dots, e_n\}$, possibly with parallel edges. Fix an arbitrary orientation \mathcal{O}_0 of G . Doing so endows G with the structure of a one-dimensional CW-complex. Thus, we obtain a short cellular chain complex

$$0 \rightarrow C_1(G; \mathbb{Q}) \xrightarrow{\partial} C_0(G; \mathbb{Q}) \rightarrow 0,$$

where $\partial(e) = v - w$ for an edge e oriented from one endpoint w to another v . We equip $C_1(G; \mathbb{Q})$ and $C_0(G; \mathbb{Q})$ with inner products by declaring that E and V form orthonormal bases for the respective chain groups. Doing so enables us to express the adjoint mapping

$$\partial^* : C_0(G; \mathbb{Q}) \rightarrow C_1(G; \mathbb{Q})$$

by the formula

$$\partial^*(v) = \sum_{e \in E} \langle \partial(e), v \rangle \cdot e.$$

The splitting

$$\text{im}(\partial^*) \oplus \ker(\partial) = C_1(G; \mathbb{Q})$$

gives rise to a pair of sublattices

$$\mathcal{C}(G) := \text{im}(\partial^*) \cap C_1(G; \mathbb{Z}) \quad \text{and} \quad \mathcal{F}(G) := \ker(\partial) \cap C_1(G; \mathbb{Z})$$

inside $C_1(G; \mathbb{Z}) \cong \mathbb{Z}^n$. These are the *cut lattice* and *flow lattice* of G , respectively. Observe that altering the choice of orientation \mathcal{O}_0 preserves the isomorphism types of $\mathcal{C}(G)$ and $\mathcal{F}(G)$.

3.2. Bases. We recall the standard construction of a pair of bases for $\mathcal{C}(G)$ and $\mathcal{F}(G)$ out of a maximal spanning forest F and orientation \mathcal{O} of G . Select an edge $e_i \in E(G)$. If $e_i \in E(F)$, then the graph $F \setminus e_i$ contains a pair of connected components K_1 and K_2 with the property that e_i directs from an endpoint in K_1 to an endpoint in K_2 in \mathcal{O} . The set of edges between K_1 and K_2 forms the *fundamental cut* $\text{cut}(F, e_i)$. We define the *cut orientation* on $\text{cut}(F, e_i)$ by directing each edge out of its endpoint in K_1 . Define

$$x_i = \sum_{v \in V(K_1)} \partial^*(v) = \sum_{e_j \in \text{cut}(F, e_i)} \epsilon_j \cdot e_j \in \mathcal{C}(G),$$

where $\epsilon_j = \pm 1$ according to whether the orientations on e_j in $\text{cut}(F, e_i)$ and \mathcal{O} agree or differ. If instead $e_i \notin E(F)$, then there exists a unique *fundamental cycle* $\text{cyc}(F, e_i)$ in $F \cup e_i$. We define the *cycle orientation* on $\text{cyc}(F, e_i)$ by orienting its edges cyclically, keeping the orientation on e_i from \mathcal{O} . Define

$$x_i = \sum_{e_j \in \text{cyc}(F, e_i)} \epsilon_j \cdot e_j \in \mathcal{F}(G),$$

where $\epsilon_j = \pm 1$ according to whether the orientations on e_j in $\text{cyc}(F, e_i)$ and \mathcal{O} agree or differ. Define a pair of sets

$$B_{\mathcal{C}} = \{x_i \mid e_i \in E(F)\} \quad \text{and} \quad B_{\mathcal{F}} = \{x_i \mid e_i \in E(G \setminus F)\}.$$

Proposition 3.1. *The cut lattice $\mathcal{C}(G)$ and flow lattice $\mathcal{F}(G)$ are complementary, primitive sublattices of $C_1(G; \mathbb{Z})$ with bases $B_{\mathcal{C}}$ and $B_{\mathcal{F}}$, respectively.*

Proof. Let $\mathcal{C}'(G) \subset \mathcal{C}(G)$ and $\mathcal{F}'(G) \subset \mathcal{F}(G)$ denote the spans of $B_{\mathcal{C}}$ and $B_{\mathcal{F}}$, respectively. Observe that if $e_i, e_j \in E(F)$, then $\langle x_i, e_j \rangle = \delta_{ij}$, while if $e_i, e_j \in E(G \setminus F)$, then $\langle x_i, e_j \rangle = \delta_{ij}$. It follows at once that $B_{\mathcal{C}}$ and $B_{\mathcal{F}}$ are bases for $\mathcal{C}'(G)$ and $\mathcal{F}'(G)$, and that $E(F)$ and $E(G \setminus F)$ evaluate on $\mathcal{C}'(G)$ and $\mathcal{F}'(G)$ precisely as the dual bases $B_{\mathcal{C}}^*$ and $B_{\mathcal{F}}^*$, respectively. Thus, $\mathcal{C}'(G)$ and $\mathcal{F}'(G)$ are primitive sublattices of $C_1(G; \mathbb{Z})$.

Now,

$$m = |B_{\mathcal{C}}| + |B_{\mathcal{F}}| = \text{rk}(\mathcal{C}'(G)) + \text{rk}(\mathcal{F}'(G)) \leq \text{rk}(\mathcal{C}(G)) + \text{rk}(\mathcal{F}(G)) \leq m,$$

where the last inequality follows since $\mathcal{C}(G)$ and $\mathcal{F}(G)$ are orthogonal. It follows that $\mathcal{C}(G)$ and $\mathcal{F}(G)$ are complementary, and furthermore that $B_{\mathcal{C}}$ and $B_{\mathcal{F}}$ are bases for the vector spaces $\mathcal{C}(G) \otimes \mathbb{Q} = \text{im}(\partial^*)$ and $\mathcal{F}(G) \otimes \mathbb{Q} = \text{ker}(\partial)$, respectively. Thus, any $x \in \mathcal{C}(G)$ has an expression

$$x = \sum_{e_i \in E(F)} q_i \cdot x_i, \quad q_i \in \mathbb{Q}.$$

However, since $q_i = \langle x, e_i \rangle \in \mathbb{Z}$, we must in fact have $x \in \mathcal{C}'(G)$. Hence $\mathcal{C}'(G) = \mathcal{C}(G)$, and similarly $\mathcal{F}'(G) = \mathcal{F}(G)$. The statement of the Lemma now follows. \square

The following Lemma ensures a particularly nice choice of spanning forest and orientation (cf. (7) in Proposition 2.8).

Lemma 3.2. *There exists a maximal spanning forest F and an orientation \mathcal{O}_1 such that*

$$\langle x_i, e_j \rangle \in \{0, 1\}, \quad \forall x_i \in B_{\mathcal{C}}, e_j \in E(G),$$

and an orientation \mathcal{O}_2 such that

$$\langle x_i, e_j \rangle \in \{0, 1\}, \quad \forall x_i \in B_{\mathcal{F}}, e_j \in E(G).$$

Proof. A *root set* R in a graph is a subset of its vertices, one in each connected component. Let R_1 be a root set of $G_1 = G$. Having defined R_i and G_i , let $G_{i+1} = G_i - R_i$, choose a root set R_{i+1} in G_{i+1} with the property that each vertex $v_{i+1} \in R_{i+1}$ has a (unique) neighbor

$v_i \in R_i$, and distinguish a single edge $e = (v_i, v_{i+1})$. Let F be the subgraph of G consisting of all such edges.

By induction on i , no vertex in R_i is contained in a cycle in F , hence F is a forest. By reverse induction on i , R_i is a root set for the subgraph of F induced on $V(G_i)$, hence $(i = 1)$ F is maximal. Given an edge $e \in E(G)$, write $e = (v_i, v_j)$, where $v_i \in R_i$, $v_j \in R_j$, and $i < j$ ($i \neq j$ since each R_i is an independent set). We obtain an orientation \mathcal{O}_1 of G by orienting each edge e from v_i to v_j , and another orientation \mathcal{O}_2 by reversing the orientation on each edge in $E(G \setminus F)$.

Observe that for all $e \in E(F)$, every edge in $\text{cut}(F, e)$ directs the same way in the cut orientation and \mathcal{O}_1 . Similarly, for all $e \in E(G \setminus F)$, every edge in $\text{cyc}(F, e)$ directs the same way in the cycle orientation and \mathcal{O}_2 . The statement of the Lemma now follows for this choice of F , \mathcal{O}_1 , and \mathcal{O}_2 . □

3.3. Short characteristic covectors. Fixing an orientation \mathcal{O}_0 , there exists a 1-1 correspondence

$$\text{Short}(C_1(G; \mathbb{Z})) \leftrightarrow \{\text{orientations } \mathcal{O} \text{ of } G\},$$

where $\chi_{\mathcal{O}} \leftrightarrow \mathcal{O}$ is determined by specifying that $\langle \chi_{\mathcal{O}}, e_i \rangle = 1$ if e_i gets the same orientation in both \mathcal{O}_0 and \mathcal{O} and -1 otherwise. The value $\langle \chi_{\mathcal{O}}, \partial^*(v) \rangle$ is thus *minus* the signed degree of v in \mathcal{O} : it equals the number of edges in D directed into v minus the number directed out of it, i.e.

$$\langle \chi_{\mathcal{O}}, \partial^*(v) \rangle = -\deg_{\mathcal{O}}(v) = \deg_{\mathcal{O}}^{\text{in}}(v) - \deg_{\mathcal{O}}^{\text{out}}(v).$$

We denote the restriction of $\chi_{\mathcal{O}}$ to $\text{Short}(\mathcal{C}(G))$ by the same symbol and call it an *orientation covector* for $\mathcal{C}(G)$.

Proposition 3.3. *The set $\text{Short}(\mathcal{C}(G))$ consists of precisely the orientation covectors for $\mathcal{C}(G)$.*

Proof. This follows in essence from a result of Hakimi [Hak65, Thm.4]; we follow the elegant treatment of Schrijver [Sch03, Thm.61.1&Cor.61.1a]. Thus, suppose that $\chi \in \text{Short}(\mathcal{C}(G))$. From $|\chi \pm 2 \sum_{v \in T} \partial^*(v)| \geq |\chi|$ we obtain

$$(18) \quad |\langle \chi, \sum_{v \in T} \partial^*(v) \rangle| \leq |\sum_{v \in T} \partial^*(v)|, \quad \forall T \subset V.$$

Define a function $l : V \rightarrow \mathbb{Z}_{\geq 0}$ by

$$l(v) = \frac{1}{2}(\deg(v) - \langle \chi, \partial^*(v) \rangle),$$

and extend l to subsets of V by declaring $l(T) = \sum_{v \in T} l(v)$. Observe that l satisfies two key properties:

$$(19) \quad l(V) = |E|, \text{ and}$$

$$(20) \quad l(T) \leq e(T), \quad \forall T \subset V,$$

where $e(T)$ denotes the number of edges with at least one endpoint in T .

We seek an orientation \mathcal{O} of G with the property that $\deg_{\mathcal{O}}^{\text{in}}(v) = l(v)$ for all $v \in V$; then $\langle \chi, \partial^*(v) \rangle = -\deg_{\mathcal{O}}(v)$, so $\chi = \chi_{\mathcal{O}}$ is an orientation covector. To produce \mathcal{O} , construct a bipartite graph B with two partite classes: V' , which contains $l(v)$ copies of v for each $v \in V$; and E , the edge set of G . The edge set of B consists of pairs (v, e) , where $v \in V'$ denotes a copy of an endpoint of $e \in E$. Properties (19) and (20) ensure that for every subset $T' \subset V'$, there exist at least $|T'|$ elements of E with a neighbor in V' . Thus, Hall's matching theorem implies that B contains a perfect matching \mathcal{M} [Sch03, Thm.16.7]. Directing each $e \in E$ to the endpoint to which it gets matched in \mathcal{M} produces the desired orientation \mathcal{O} . \square

Corollary 3.4. *The restriction maps*

$$\text{Short}(C_1(G; \mathbb{Z})) \rightarrow \text{Short}(\mathcal{C}(G)), \text{Short}(\mathcal{F}(G))$$

surject, and the inclusion $\mathcal{C}(G) \oplus \mathcal{F}(G) \subset C_1(G; \mathbb{Z})$ induces a natural isomorphism

$$\varphi : (C(\mathcal{C}(G)), d_{\mathcal{C}}) \xrightarrow{\sim} (C(\mathcal{F}(G)), -d_{\mathcal{F}}).$$

Proof. This follows immediately on combination of Propositions 2.7, 3.1, and 3.3. \square

3.4. Whitney's theorem. Now suppose that G is connected (by convention, the empty graph is connected). A *cut-edge* $e \in E(G)$ is one such that $G - e$ is disconnected, and a *cut-vertex* $v \in V(G)$ is one such that $G - v$ is disconnected. The graph G is *2-edge-connected* if it does not contain a cut-edge and *2-connected* if it does not contain a cut-vertex. It is straightforward to show that G is 2-edge-connected iff every edge is contained in some cycle, and 2-connected iff every pair of distinct edges is contained in some cycle. Thus, a 2-connected graph is 2-edge-connected, and the graph with one vertex and no edge is 2-edge-connected. A *2-isomorphism* between a pair of graphs is a cycle-preserving bijection between their edge sets.

A special instance of 2-isomorphism arises as follows. Let G_1, G_2 denote a pair of disjoint graphs, and distinguish a pair of distinct vertices $v_i, w_i \in V(G_i)$, $i = 1, 2$. Form a graph G by identifying the vertices v_1, v_2 into a vertex v and vertices w_1, w_2 into a vertex w ; and similarly, form a graph G' by identifying the vertices v_1, w_2 into a vertex v' and vertices w_1, v_2 into a vertex w' . We say that G and G' are related by a *switch*. The switch is *special* if one of v_i, w_i is an isolated vertex in G_i for some i . In this case, one of v, w is a cut-vertex in G and one of v', w' is a cut-vertex in G' . It is clear that identifying $E(G_i) \subset E(G)$ with $E(G_i) \subset E(G')$, $i = 1, 2$, defines a 2-isomorphism between G and G' .

Conversely, we have the following important fact.

Theorem 3.5 (Whitney [Whi33]). *A pair of 2-connected graphs are 2-isomorphic iff they are related by a sequence of switches.* \square

Truemper gave a short, simple proof of Theorem 3.5 [Tru80].

We now develop a straightforward generalization of Theorem 3.5 to the case of an arbitrary connected graph G . A *block* $B \subset G$ is a maximal 2-connected subgraph of G . In particular,

the cut-edges of G constitute its 1-edge blocks, and the cut-vertices of G are the vertices of intersection between distinct blocks of G . Let $T(G)$ denote the set of edges contained in some cycle in G ; thus, $e \in T(G)$ iff e is not a cut-edge. A *2-isomorphism* between a pair of connected graphs G, G' is a cycle-preserving bijection between $T(G)$ and $T(G')$.

Given a cut-edge $e \in E(G)$, we contract it to obtain a new graph G/e . We say that G/e is obtained from G by *cut-edge contraction*, and conversely that G is obtained from G/e by *cut-edge expansion*. It is clear that both cut-edge contraction and expansion preserve the 2-isomorphism type of a graph.

With these definitions in place, we state the desired generalization of Theorem 3.5.

Proposition 3.6. *A pair of connected graphs are 2-isomorphic iff they are related by a sequence of switches and cut-edge contractions and expansions. Furthermore, only switches are necessary if the graphs are 2-edge-connected.*

Proof. For the first part, we just need to establish the forward implication. Write $H \approx H'$ if H is related to H' by a sequence of switches and cut-edge contractions and expansions.. Clearly, \approx defines an equivalence relation on graphs. Now, suppose that G and G' are a pair of 2-isomorphic, connected graphs. In each graph, contract all the cut-edges and perform a sequence of special switches so that there is a vertex in common to all remaining blocks (it will be the unique cut-vertex if there are multiple blocks). The resulting graphs $G_0 \approx G$ and $G'_0 \approx G'$ are 2-isomorphic by some mapping φ . Put an equivalence relation \sim on $E(H)$ by declaring $e \sim f$ if $e = f$ or e and f belong to some cycle. Thus, the edge sets of blocks of H are precisely the equivalence classes under \sim . Since φ clearly preserves \sim , it follows that φ pairs the blocks of G_0 and G'_0 , and furthermore defines a 2-isomorphism between each such pair (B_0, B'_0) . By Theorem 3.5, it follows that B_0 and B'_0 are related by a sequence of switches. Each switch in B_0 extends to a switch in G_0 , the composition of which results in a graph $\overline{G} \approx G_0$ whose blocks are isomorphic in pairs with those of G'_0 . A sequence of special switches now transforms \overline{G} into G'_0 . Thus, $G \approx G_0 \approx \overline{G} \approx G'_0 \approx G'$, as desired. Lastly, if G and G' are 2-edge-connected, then $G = G_0$ and $G' = G'_0$, so only switches are necessary to establish $G \approx G'$. □

3.5. Graph lattices with the same d -invariant. For a pair of lattices Λ_1, Λ_2 , write $\Lambda_1 \simeq \Lambda_2$ if $\Lambda_1 \oplus \mathbb{Z}^k \cong \Lambda_2$ or $\Lambda_1 \cong \Lambda_2 \oplus \mathbb{Z}^k$ for some k . The following Proposition and its proof are essentially due to Bacher, et al. [BdlHN97, Prop.5].

Proposition 3.7. *If G and G' are 2-isomorphic, then $\mathcal{F}(G) \cong \mathcal{F}(G')$ and $\mathcal{C}(G) \simeq \mathcal{C}(G')$.*

Proof. By Proposition 3.6, it suffices to establish the statement of the Proposition under the assumption that G' is obtained from G by a switch or a cut-edge contraction.

Suppose first that $G' = G/e$ for a cut-edge $e \in E(G)$. An orientation on G induces one on G' and identifies e with a basis element in $C_1(G; \mathbb{Z})$. We obtain a natural isomorphism between $e^\perp \subset C_1(G; \mathbb{Z})$ and $C_1(G'; \mathbb{Z})$. This isomorphism clearly carries $\mathcal{F}(G) \subset e^\perp$ onto $\mathcal{F}(G')$ and $\mathcal{F}(G)^\perp \cap e^\perp$ onto $\mathcal{F}(G')^\perp = \mathcal{C}(G')$. Note as well that $\mathcal{C}(G) = \mathcal{F}(G)^\perp = (\mathcal{F}(G)^\perp \cap e^\perp) \oplus (e) \cong \mathcal{C}(G') \oplus \mathbb{Z}$. Thus, $\mathcal{F}(G) \cong \mathcal{F}(G')$ and $\mathcal{C}(G) \simeq \mathcal{C}(G')$.

Next, suppose that G and G' are related by a switch. Choose an orientation \mathcal{O}_i of G_i , $i = 1, 2$, and let $\mathcal{O}, \mathcal{O}'$ denote the induced orientations of G, G' . Doing so leads to natural isomorphisms

$$C_1(G_1; \mathbb{Z}) \oplus C_1(G_2; \mathbb{Z}) \xrightarrow{\sim} C_1(G; \mathbb{Z}), \quad (x, y) \mapsto x + y$$

and

$$C_1(G_1; \mathbb{Z}) \oplus C_1(G_2; \mathbb{Z}) \xrightarrow{\sim} C_1(G'; \mathbb{Z}), \quad (x, y) \mapsto x - y.$$

Define

$$\tilde{\mathcal{F}}(G_i) = \{x \in C_1(G_i; \mathbb{Z}) \mid \langle \partial x, u \rangle = 0, \forall u \neq v_i, w_i\}, \quad i = 1, 2.$$

Observe that $\text{im}(\partial)$ is contained in the kernel of the augmentation map $C_0 \rightarrow \mathbb{Z}$ defined by sending all vertices to 1. It follows that $x \in \tilde{\mathcal{F}}(G_i)$ satisfies

$$(21) \quad \langle \partial x, v_i \rangle + \langle \partial x, w_i \rangle = 0, \quad i = 1, 2.$$

Thus, the preceding isomorphisms restrict to isomorphisms

$$\{(x, y) \in \tilde{\mathcal{F}}(G_1) \oplus \tilde{\mathcal{F}}(G_2) \mid \langle \partial x, v_1 \rangle + \langle \partial y, v_2 \rangle = 0\} \xrightarrow{\sim} \mathcal{F}(G)$$

and

$$\{(x, y) \in \tilde{\mathcal{F}}(G_1) \oplus \tilde{\mathcal{F}}(G_2) \mid \langle \partial x, w_1 \rangle - \langle \partial y, v_2 \rangle = 0\} \xrightarrow{\sim} \mathcal{F}(G'),$$

respectively. By (21), the two sublattices of $\tilde{\mathcal{F}}(G_1) \oplus \tilde{\mathcal{F}}(G_2)$ appearing here coincide. Thus, $\mathcal{F}(G) \cong \mathcal{F}(G')$. Since this isomorphism is induced by the isomorphism $C_1(G; \mathbb{Z}) \cong C_1(G'; \mathbb{Z})$, it follows that their orthogonal complements are isomorphic as well: $\mathcal{C}(G) \cong \mathcal{C}(G')$. This completes the proof of the Proposition. \square

We now state our main combinatorial result, an extension of Theorem 1.2.

Theorem 3.8. *The following assertions are equivalent for a pair of connected graphs G, G' :*

- (1) G and G' are 2-isomorphic;
- (2) $\mathcal{F}(G) \cong \mathcal{F}(G')$;
- (3) $\mathcal{C}(G) \simeq \mathcal{C}(G')$;
- (4) the d -invariants of $\mathcal{F}(G)$ and $\mathcal{F}(G')$ are isomorphic; and
- (5) the d -invariants of $\mathcal{C}(G)$ and $\mathcal{C}(G')$ are isomorphic.

Proof. (1) \implies (2), (3) follow from Proposition 3.7; (2) \implies (4) and (3) \implies (5) are clear (using additivity of the d -invariant in the second case); and (4) \iff (5) follows from Corollary 3.4. We proceed to establish (5) \implies (1), from which the Theorem follows.

Note that contracting all the cut-edges in a connected graph results in a 2-edge-connected graph with the same 2-isomorphism type. Hence it suffices to establish (5) \implies (1) under the assumption that both graphs are 2-edge-connected. Thus, suppose that $(C(\mathcal{C}(G)), d_{\mathcal{C}}) \cong (C(\mathcal{C}(G')), d'_{\mathcal{C}})$ for a pair of 2-edge-connected graphs G, G' . By Corollary 3.4, there exists a natural isomorphism $(C(\mathcal{C}(G')), d'_{\mathcal{C}}) \xrightarrow{\sim} (C(\mathcal{F}(G')), -d'_{\mathcal{F}})$. Consequently, we obtain an isomorphism

$$\varphi : (C(\mathcal{C}(G)), d_{\mathcal{C}}) \xrightarrow{\sim} (C(\mathcal{F}(G')), -d'_{\mathcal{F}}).$$

Proposition 2.7 implies that the glue lattice

$$Z_2 := \mathcal{C}(G) \oplus_{\varphi} \mathcal{F}(G')$$

admits an orthonormal basis, and furthermore that the restriction maps

$$\text{Short}(Z_2) \rightarrow \mathcal{C}(G), \mathcal{F}(G')$$

surject.

Now, let (Λ, Z_1) denote either pair $(\mathcal{C}(G), C_1(G; \mathbb{Z}))$ or $(\mathcal{F}(G'), C_1(G'; \mathbb{Z}))$. By Corollary 3.4, the restriction map $\text{Short}(Z_1) \rightarrow \Lambda$ surjects, and by Lemma 3.2, Z_1 admits an orthonormal basis B_1 such that (7) holds. Every edge of G is contained in some cut, and by 2-edge-connectivity, every edge of G' is contained in some cycle. It follows in either case that (8) holds. Thus, Proposition 2.8 applies and furnishes embeddings ι, ι' such that the following diagram commutes:

$$(22) \quad \begin{array}{ccccc} C_1(G; \mathbb{Z}) & \xhookrightarrow{\iota} & Z_2 & \xleftarrow{\iota'} & C_1(G'; \mathbb{Z}) \\ & \nwarrow \iota_1 & \nearrow \iota_2 & \nwarrow \iota'_2 & \nearrow \iota'_1 \\ & & \mathcal{C}(G) & & \mathcal{F}(G') \end{array}$$

Switching the roles of G and G' , we can repeat the same construction with respect to the glue lattice

$$Z'_2 := \mathcal{C}(G') \oplus_{\psi} \mathcal{F}(G),$$

using an isomorphism

$$\psi : (C(\mathcal{C}(G')), d'_C) \xrightarrow{\sim} (C(\mathcal{F}(G)), -d_F).$$

We obtain two more embeddings from Proposition 2.8, leading to a total of four inequalities

$$(23) \quad \text{rk}(C_1(G; \mathbb{Z})), \text{rk}(C_1(G'; \mathbb{Z})) \leq \text{rk}(Z_2), \text{rk}(Z'_2).$$

On the other hand,

$$\begin{aligned} \text{rk}(C_1(G; \mathbb{Z})) + \text{rk}(C_1(G'; \mathbb{Z})) &= \text{rk}(\mathcal{C}(G)) + \text{rk}(\mathcal{F}(G)) + \text{rk}(\mathcal{C}(G')) + \text{rk}(\mathcal{F}(G')) \\ &= \text{rk}(Z_2) + \text{rk}(Z'_2). \end{aligned}$$

Hence each inequality (23) is an equality, so the embeddings ι, ι' are actually isomorphisms. Thus, we obtain a composite isomorphism

$$f := (\iota')^{-1} \circ \iota : C_1(G; \mathbb{Z}) \xrightarrow{\sim} C_1(G'; \mathbb{Z}),$$

and restricting f to the orthonormal bases induces a bijection

$$f_E : E(G) \xrightarrow{\sim} E(G').$$

We claim that f_E is a 2-isomorphism. First note that (22) and Propositions 2.7 and 3.1 show that f carries $\mathcal{F}(G)$ isomorphically onto $\mathcal{F}(G')$. Now let C denote a cycle in G . With an orientation \mathcal{O} of G fixed and an arbitrary edge $e \in E(C)$ distinguished, we obtain an element $x(C) \in \mathcal{F}(G) \subset C_1(G; \mathbb{Z})$ as in §3.2. Thus we obtain an element $f(x(C)) \in \mathcal{F}(G') \subset C_1(G'; \mathbb{Z})$ with the property that

$$|\langle f(x(C)), e' \rangle| \in \{0, 1\}, \quad \forall e' \in C_1(G'; \mathbb{Z}), |e'| = 1.$$

With an orientation \mathcal{O}' of G' fixed, it follows that the subgraph $f_E(E(C))$ is an (oriented) Eulerian subgraph of G' . Hence it decomposes into an edge-disjoint union of directed cycles. Choose one and denote it by C' . By symmetry, f_E^{-1} carries C' onto a non-empty Eulerian subgraph of C ; but since C is a cycle, it follows that $f_E^{-1}(E(C')) = E(C)$. Hence f_E carries the cycle C to the cycle C' . Since C was arbitrary, it follows that f_E is a 2-isomorphism, as claimed. \square

4. CONWAY MUTATION.

4.1. Planar graphs. By abuse of terminology, we regard a plane drawing Γ of a planar graph G as an embedding in the sphere $S^2 = \mathbb{R}^2 \cup \{\infty\}$.

Connectivity properties of G are reflected by the topology of Γ in the following way. Suppose that $\{v, w\}$ is a cut-set in G , where $v, w \in V(G)$ need not be distinct. Then there exists a circle $S^1 \subset S^2$ such that $S^1 \cap \Gamma = \{v, w\}$ and both components of $S^2 - S^1$ contain a vertex of G . Conversely, given such a circle with $S^1 \cap \Gamma = \{v, w\}$, it follows that $\{v, w\}$ is a cut-set in G .

Choose either disk bounded by S^1 and reglue it by an orientation-reversing homeomorphism that fixes v and w . Doing so results in another plane drawing Γ' of G , and we say that Γ, Γ' differ by a *flip*. Conversely, we have the following result.

Proposition 4.1 (Mohar-Thomassen [MT01], Thm.2.6.8¹). *Any two plane drawings of a 2-connected planar graph G are related by a sequence of flips and isotopies in the sphere.* \square

Alternatively, choose either disk bounded by S^1 and reglue it by a homeomorphism that exchanges v and w . Doing so results in a plane drawing Γ' of a graph G' related to G by a switch. We say that Γ, Γ' differ by a *planar switch*. The planar switch is *positive* or *negative* according to whether the homeomorphism preserves or reverses orientation. Note that a positive and negative planar switch differ by composition with a flip.

Lastly, suppose that there exists a pair of disks D_1^2, D_2^2 such that $D_1^2 \cap \Gamma = \Gamma \cap D_2^2 = D_2^2 \cap D_1^2 = \{v\}$ for some $v \in V(\Gamma)$. Exchange D_1^2 and D_2^2 by a homeomorphism that preserves v ; doing so results in another plane drawing Γ' of G , and we say that Γ, Γ' differ by a *swap*. The swap is *positive* or *negative* according to whether the homeomorphism preserves or reverses orientation.

Examples of flips, planar switches, and swaps appear in Figures 2-4.

Lemma 4.2. *Any two plane drawings Γ, Γ' of a connected planar graph G are related by a sequence of flips, swaps, and isotopies.*

Proof sketch. If G is 2-connected, then Proposition 4.1 applies at once, so suppose otherwise and fix a cut-vertex $v \in V(G)$. Decompose G uniquely into a maximal collection of subgraphs G_1, \dots, G_k that intersect pairwise in v . For each i , let $\Gamma_i \subset \Gamma, \Gamma'_i \subset \Gamma'$ denote the induced

¹As remarked on [MT01, p.3], the results of that book are stated for *simple* graphs, i.e. those without parallel edges, but most results (including this one) apply, *mutatis mutandis*, to graphs with parallel edges.

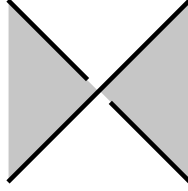


FIGURE 1. Coloring convention for alternating diagrams.

plane drawings of G_i . Reindexing the subgraphs if necessary, there exists a sequence of disks $D_1^2, \dots, D_k^2 \subset S^2$ whose boundaries intersect pairwise in v and such that

$$\Gamma_i = \Gamma \cap (D_i^2 \setminus \text{int}(\bigcup_{j=1}^{i-1} D_j^2)), \quad \forall i.$$

A sequence of at most $k-2$ swaps results in a plane drawing $\bar{\Gamma}$ of G such that there exist disks $\bar{D}_1^2, \dots, \bar{D}_k^2$ that intersect pairwise in v and satisfy $\bar{\Gamma}_i = \bar{\Gamma} \cap \bar{D}_i^2$. Similarly, a sequence of swaps and isotopies transforms Γ' into a plane drawing $\bar{\Gamma}'$ of G such that $\bar{\Gamma}'_i = \bar{\Gamma}' \cap \bar{D}_i^2$ for this same collection of disks. By induction on $|V|$, there exists a sequence of flips, swaps, and isotopies supported in \bar{D}_i^2 that transforms $\bar{\Gamma}_i \subset S_i^2 := \bar{D}_i^2 / \partial \bar{D}_i^2$ into $\bar{\Gamma}'_i \subset S_i^2$. This sequence, together with another sequence of swaps in S^2 , transforms $\bar{\Gamma}$ into $\bar{\Gamma}'$. Thus, Γ and Γ' are related in the desired manner. □

Corollary 4.3. *Let Γ, Γ' denote plane drawings of a pair of 2-isomorphic, 2-edge-connected planar graphs G, G' . Then Γ and Γ' are related by a sequence of flips, planar switches, swaps, and isotopies.*

Proof. Suppose first that G, G' are related by a single switch. In this case, there clearly exists a planar switch of Γ that results in a plane drawing Γ'_0 of G' . By Lemma 4.2, there exists a sequence of flips, swaps, and isotopies that transforms Γ'_0 into Γ' . The general case of the Corollary now follows from Proposition 3.6. □

4.2. Between diagrams and graphs. Let $D \subset S^2$ denote a connected, alternating diagram for a link L . Color the regions of D black and white in checkerboard fashion according to the coloring convention displayed in Figure 1. We obtain a planar graph by drawing a vertex in each black region and an edge for each crossing that joins a pair of black regions. Examples appear in Figures 2-4. The result is the *Tait graph* G of D , equipped with a natural (isotopy class of) plane drawing Γ . This process is clearly reversible: given a connected plane drawing Γ , we obtain from it a connected, alternating link diagram D .

For concreteness, write $S^3 = \mathbb{R}^3 \cup \{\infty\}$, where \mathbb{R}^3 has coordinates x, y, z , and let $S^2 \subset S^3$ denote the xy -coordinate plane together with ∞ . Suppose that the unit ball $B^3 \subset \mathbb{R}^3$ meets L in the four points $\{(\pm 1/\sqrt{2}, \pm 1/\sqrt{2}, 0)\}$, and these are all regular points for D . The sphere

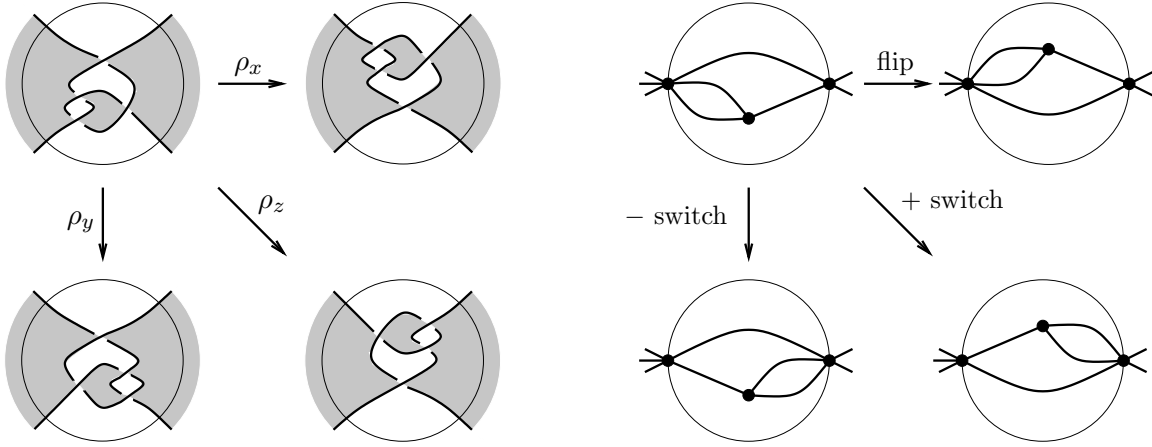


FIGURE 2. Between mutations and graph operations (1).

∂B^3 is a *Conway sphere* for L , and the circle $\partial B^3 \cap S^2$ is a *Conway circle* for D . More generally, a Conway circle for D refers to any circle $S^1 \subset S^2$ meeting D in four regular points; by a suitable isotopy, we can arrange that S^1 arises in the manner just described. We operate on $B^3 \cap L$ by performing a 180° rotation about one of the three coordinate axes. The result is a link L' and a corresponding diagram $D' \subset S^2$. We say that the links L, L' differ by a (*Conway*) *mutation*, and a pair of links are *mutants* if they differ by a sequence of isotopies and mutations. We make similar definitions at the level of diagrams, requiring all isotopies to take place in S^2 . Thus, mutant diagrams present mutant links, but the converse does not hold in general.

Lemma 4.4. *Let D denote a connected, alternating diagram and Γ the associated plane drawing of its Tait graph. A mutation of D effects one or two flips, a planar switch, or a swap in Γ . Conversely, a flip, planar switch, or swap in Γ effects a mutation of D .*

Proof. Draw D and Γ simultaneously in S^2 , and choose a Conway circle for D with respect to which to mutate. Choose coordinates so that the Conway circle arises in the concrete manner described above, and let $D_0^2, D_\infty^2 \subset S^2$ denote the two disks that it bounds. By rotating the diagram 90° if necessary, we may assume that the points $(\pm 1, 0, 0)$ lie in black regions of D .

For the forward implication, we distinguish two cases, depending on whether or not $(-1, 0, 0)$ belongs to the same black region as $(1, 0, 0)$ or not. Suppose first that it belongs to a different black region. By an isotopy of Γ , we may assume that the points $(\pm 1, 0, 0)$ represent distinct vertices $v, w \in V(\Gamma)$. Let $\Gamma_1 \subset \Gamma$ denote the subgraph induced on the regions of D_0^2 and $\Gamma_2 \subset \Gamma$ the subgraph induced on the regions of D_∞^2 . By inspection, rotation of the unit disk about the x -axis corresponds to a flip of $\Gamma_1 \subset \Gamma$; rotation about the y -axis corresponds to a negative planar switch; and rotation about the z -axis corresponds to a positive planar switch (Figure 2). This establishes the forward implication of the Lemma in this case.

Suppose instead that $(-1, 0, 0)$ belongs to the same black region as $(1, 0, 0)$. By an isotopy of D , and possibly a change of coordinates that exchanges D_0^2 and D_∞^2 , we may assume that

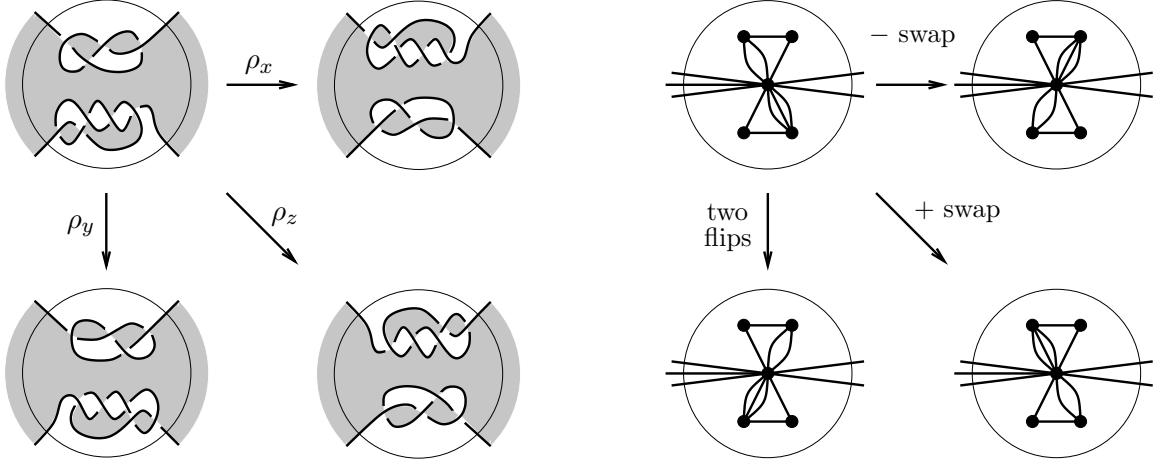


FIGURE 3. Between mutations and graph operations (2).

the interval $\{(t, 0, 0), -1 \leq t \leq 1\}$ is supported in this black region. Thus, the diagram meets the unit disk in a split pair of (possibly knotted) strands. By an isotopy of Γ , we may assume that $(0, 0, 0)$ represents a vertex $v \in V(\Gamma)$. Let $\Gamma_1 \subset \Gamma$ denote the subgraph induced on the regions of $D_0^2 \cap \{y \geq 0\}$ and $\Gamma_2 \subset \Gamma$ the subgraph induced on the regions of $D_0^2 \cap \{y \leq 0\}$. By inspection, rotation of the unit disk about the x -axis corresponds to a negative swap of $\Gamma_1, \Gamma_2 \subset \Gamma$; rotation about the y -axis corresponds to flipping both Γ_1 and Γ_2 ; and rotation about the z -axis corresponds to positive swap (Figure 3). This establishes the forward implication of the Lemma in this case.

For the reverse direction, we distinguish several cases as well. First consider the case of a flip or a planar switch (positive or negative) that involves a pair of distinct vertices $v, w \in V(\Gamma)$. In each case, there exists a circle $S^1 \subset S^2$ such that $S^1 \cap \Gamma = \{v, w\}$, and it is a Conway circle for D . Just as in the first case of the forward implication considered above, a flip and the two types of planar switch correspond to three types of Conway mutation with respect to this circle.

Next consider the case of a swap involving a vertex v . In this case, the boundary of a regular neighborhood of the disks D_1^2, D_2^2 involved is a Conway circle for D . By an isotopy we may arrange so that the disk D_0^2 bounded by this circle is the unit disk and $\{(t, 0, 0), -1 \leq t \leq 1\}$ is supported in the black region that contains v . Now a positive swap corresponds to rotating about the z -axis, while a negative swap corresponds to rotating about the x -axis. In either case we obtain a mutation.

Lastly, consider the case of a flip involving a single vertex v . In this case, the circle $S^1 \subset S^2$ involved in the flip meets D in a pair of points. Apply an isotopy of D as in Figure 4 to introduce two new intersection points with S^1 . A rotation about the x -axis followed by an isotopy effects a rotation in the y -axis in the original disk, and this corresponds to the flip. Thus we obtain a mutation in this case as well.

□

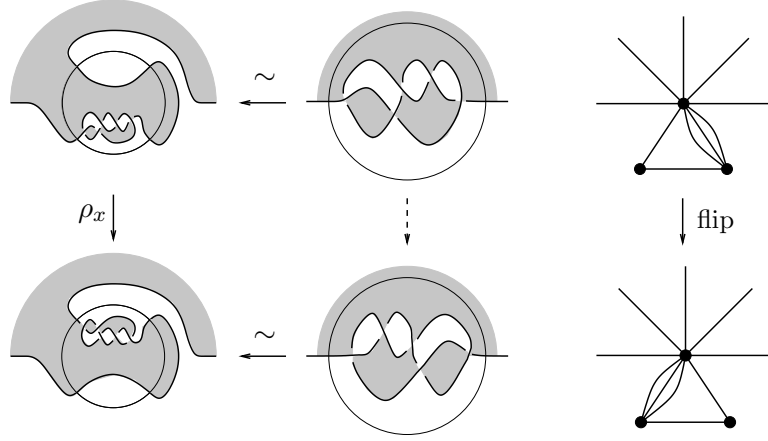


FIGURE 4. Between mutations and graph operations (3).

Corollary 4.5 (cf. [CK08], §3, Prop.1). *A pair of connected, reduced alternating diagrams are mutants iff their Tait graphs are 2-isomorphic.*

Proof. This follows at once from Corollary 4.3 and Lemma 4.4, noting that a diagram is connected and reduced iff its Tait graph is 2-edge-connected. \square

4.3. Two-bridge links. We apply Corollary 4.5 to derive the result concerning two-bridge links quoted in §1.4.

Proposition 4.6. *A pair of 2-bridge link diagrams in standard position are mutants iff they coincide up to isotopy and reversal (i.e. an orientation-reversing homeomorphism of S^2).*

Proof. First observe that the Proposition is obvious in the case that one of the diagrams is the standard diagram for the unknot or two-component unlink. Excluding these cases, let D denote a 2-bridge link diagram in standard position. This means that D is connected, reduced, and alternating; its Tait graph G contains a Hamiltonian cycle H ; every edge in $E(G) - E(H)$ is incident with a fixed vertex $v_0 \in V(G)$; and in the plane drawing Γ of G , the interiors of all these edges lie in one fixed region of $S^2 - H$. Note that these conditions specify the image of Γ up to isotopy and reversal.

Let $e, f \in E(H)$ denote the edges incident with v_0 , and let v_1, \dots, v_k denote the neighbors of v_0 in $G - \{e, f\}$, chosen with respect to some cyclic order on H (thus, $k = 0$ iff $G = H$). Let E_i denote the set of edges between v_0 and v_i in $G - \{e, f\}$, for $i = 0, \dots, k+1$, and let F_i denote the edge set of the path directed from v_i to v_{i+1} along H , for $i = 0, \dots, k$. Here we take $v_{k+1} = v_0$, so $E_0 = E_{k+1} = \emptyset$, and $F_0 = E(H)$ iff $k = 0$.

Now suppose that D' is another 2-bridge link diagram in standard position, and it is a mutant of D . By Corollary 4.5, there exists a 2-isomorphism $\varphi : E(G) \rightarrow E(G')$, where G' denotes the Tait graph of D' . We argue that, in fact, $G \cong G'$.

Since G' is 2-edge-connected, it follows that $\varphi(E(H))$ is the edge set of a Hamiltonian cycle H' in G' . Define $v'_0, \dots, v'_{k'+1}, E'_0, \dots, E'_{k'+1}, F'_0, \dots, F'_{k'}$ with respect to G' as above. Observe that the edge sets \emptyset and $\bigcup_{t=i}^j F_t$, $i \leq j$, are the intersections between cycles of G with $E(H)$. Also, any cycle that meets H in F_i uses an edge from E_i and an edge from E_{i+1} . The same applies to G' , mutatis mutandis. It follows that φ carries F_0, \dots, F_k to $F'_0, \dots, F'_{k'}$ in (possibly reverse) order, and E_0, \dots, E_k to $E'_0, \dots, E'_{k'}$ in the same order. In particular, $k = k'$. Since the E 's are sets of parallel edges and the F 's are edge sets of paths, it follows that $G \cong G'$. (Note, however, that this isomorphism does not necessarily induce φ .)

Since $G \cong G'$ and the images of Γ, Γ' are unique up to isotopy and reversal, it follows that D and D' coincide up to isotopy and reversal as well. \square

4.4. Heegaard Floer homology. We recall here the necessary input from Heegaard Floer homology. We work with the simplest version of this theory, namely the invariant $\widehat{HF}(Y)$, defined over the field \mathbb{F}_2 , for a rational homology sphere Y . The invariant takes the form of a finite-dimensional vector space over \mathbb{F}_2 , graded by spin^c structures on Y and rational numbers:

$$\widehat{HF}(Y) = \bigoplus_{\mathfrak{s} \in \text{Spin}^c(Y)} \widehat{HF}(Y, \mathfrak{s}), \quad \widehat{HF}(Y, \mathfrak{s}) = \bigoplus_{d \in \mathbb{Q}} \widehat{HF}_d(Y, \mathfrak{s}).$$

Spin^c structures on Y form a torsor over the group $H^2(Y; \mathbb{Z})$. We may therefore regard the invariant as a pair $(\text{Spin}^c(Y), \widehat{HF}(Y, \cdot))$, where $\widehat{HF}(Y, \cdot)$ takes values in the set of finitely generated, rationally graded vector spaces over \mathbb{F}_2 . From this point of view, a pair of rational homology spheres Y_1, Y_2 have isomorphic Heegaard Floer homology groups if there exists an isomorphism $(\text{Spin}^c(Y_1), \widehat{HF}(Y_1, \cdot)) \xrightarrow{\sim} (\text{Spin}^c(Y_2), \widehat{HF}(Y_2, \cdot))$.

The invariant has a fundamental non-vanishing property: $\widehat{HF}(Y, \mathfrak{s}) \neq 0$, $\forall \mathfrak{s} \in \text{Spin}^c(Y)$. Furthermore, there exists a distinguished grading, the *correction term* $d(Y, \mathfrak{s}) \in \mathbb{Q}$, with the property that $\widehat{HF}_{d(Y, \mathfrak{s})}(Y, \mathfrak{s}) \neq 0$. By definition, the pair $(\text{Spin}^c(Y), d(Y, \cdot))$ is the *d-invariant* of Y . The space Y is an *L-space* if $\text{rk } \widehat{HF}(Y, \mathfrak{s}) = 1$, $\forall \mathfrak{s} \in \text{Spin}^c(Y)$. In this case, $\widehat{HF}(Y, \mathfrak{s})$ is supported in the single grading $d(Y, \mathfrak{s})$. Thus, for an L-space Y , the isomorphism type of its Heegaard Floer homology groups determines and is determined by that of its *d-invariant*.

Theorem 4.7 (Ozsváth-Szabó [OSz05], Thm.3.4). *Let L denote a non-split alternating link and G the Tait graph of an alternating diagram of L . The branched double-cover $\Sigma(L)$ is an L-space, and the (Heegaard Floer) *d-invariant* of $\Sigma(L)$ is isomorphic to minus the (lattice theoretic) *d-invariant* of $\mathcal{F}(G)$.* \square

We arrive at last to the proof of our main topological result.

Proof of Theorem 1.1. (1) \implies (2) is clear, (2) \implies (3) is the observation of Viro, and (3) \implies (4) is clear. It stands to establish (4) \implies (1). By Theorem 4.7, the *d-invariants* of $\mathcal{F}(G)$ and $\mathcal{F}(G')$ are isomorphic, where G, G' denote the Tait graphs of D, D' . By Theorem 3.8, it follows that G and G' are 2-isomorphic, so by Corollary 4.5, it follows that D and D' are mutants. This completes the proof of the Theorem. \square

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